Discontinuous Galerkin Methods for Periodic Boundary Value Problems

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This article considers the extension of well-known discontinuous Galerkin (DG) finite element formulations to elliptic problems with periodic boundary conditions. Such problems routinely appear in a number of applications, particularly in homogenization of composite materials. We propose an approach in which the periodicity constraint is incorporated weakly in the variational formulation of the problem. Both $H^1$ and $L^2$ error estimates are presented. A numerical example confirming theoretical estimates is shown. © 2006 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 23: 000–000, 2007

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I. INTRODUCTION

Boundary value problems with periodic boundary conditions appear in a number of applications, particularly in the homogenization of composite materials with a periodic microstructure [1, 2]. When such problems are solved numerically, the periodicity condition is often imposed strongly, i.e., the solution values on periodic edges are required to match exactly.

In this article, we propose a weak imposition of periodicity within the discontinuous Galerkin finite element framework. Such an approach is meaningful because the DG framework naturally permits the solution to be discontinuous across element edges, whether internal or periodic. Also, since periodic boundary value problems arise in situations where a unit cell is repeated indefinitely, the use of weak periodicity amounts to treating the unit cell itself as a discontinuous super-element. The crux of the proposed approach is the treatment of periodic edges as interior edges with appropriate definitions of the so-called “jump” and “average” terms that are common in DG methods.

We discuss three DG formulations for periodic boundary value problems: the periodic nonsymmetric interior penalty Galerkin (PNIPG) method, the periodic symmetric interior penalty Galerkin (PSIPG) method, and the periodic discontinuous Galerkin method (PDG). Nonperiodic versions of each of these formulations have been already been discussed in the literature.
The nonsymmetric interior penalty Galerkin (NIPG) method was put forward by Rivière et al. [3], the PSIPG method by Wheeler [4] and Arnold [5], and the discontinuous Galerkin (DG) method by Baumann, Oden and Babuška [6, 7].

In what follows, we first introduce the notation and a model boundary value problem with periodic boundary conditions and then three DG formulations of the problem. Next, we present without proof a priori error estimates for each formulation. Finally, a numerical example is discussed, followed by some concluding remarks.

II. DEFINITIONS

Partition
Without loss of generality, assume the periodic boundary value problem is posed on a domain \( \Omega = (0, 1)^N \), with \( N = 2 \) or \( 3 \). Consider a partition \( \mathcal{P} \) of the domain \( \Omega \) into \( M \) nonoverlapping finite elements \( \omega_k \) such that

\[
\omega_i \cap \omega_j = \emptyset \quad \text{for} \quad i \neq j, \quad \Omega = \bigcup_{k=1}^{M} \omega_k. \tag{2.1}
\]

Let \( \partial \omega_k \) denote the boundary of element \( k \). Let \( h_k \) be the diameter of element \( k \) and let \( h = \max\{h_k, \, k = 1, 2, \ldots, M\} \).

Let there be \( N_i \) interior edges in the mesh and \( N_b \) boundary edge-pairs. The interior edges of the partition are denoted \( \{e_k\}, \, k = 1, 2, \ldots, N_i \). If \( e_k \) is an interior edge between elements \( i \) and \( j, \, i > j \), then the unit normal \( n \) associated with the edge points from element \( i \) to element \( j \).

The boundary edge-pairs are denoted \( \gamma_k = \{\gamma^+_k, \gamma^-_k\}, \, k = 1, 2, \ldots, N_b \) (Fig. 1). If \( \gamma^+_k \subset \partial \omega_l \) and \( \gamma^-_k \subset \partial \omega_m \) with \( l > m \), then we associate with the edge-pair \( \gamma_k \) a normal \( n \) that is directed away from \( \gamma^+_k \) and into \( \gamma^-_k \), as shown in Fig. 1.

Function Spaces
We define the so-called broken space on \( \Omega \) as follows:

\[
H^s(\mathcal{P}) = \{ v \in L^2(\Omega), \, v|_{\omega_k} \in H^s(\omega_k), \, 1 \leq k \leq M \}. \tag{2.2}
\]

Associated with this space, we define a norm:

\[
\|v\|_{s, \mathcal{P}} = \sqrt{\sum_{k} \|v\|^2_{H^s(\omega_k)}} \tag{2.3}
\]

where \( \| \cdot \|_{H^s(\omega_k)} \) is the usual Sobolev norm.

The finite element space is given by

\[
V^p = \prod_{k=1}^{M} P_p(\omega_k), \tag{2.4}
\]

where \( p \) is a positive integer and \( P_p(\omega_k) \) is the set of polynomials of degree less than or equal to \( p \) on \( \omega_k \).
Jump and Average Operators

For each interior edge $e_k$, we can define the jump $[\cdot]$ and average $\langle \cdot \rangle$ of a function $v \in H^s(P)$, $s > 1/2$, as follows:

$$[v]|_{e_k} = v|_{\omega_i} - v|_{\omega_j}, \quad \langle v \rangle|_{e_k} = \frac{1}{2}(v|_{\omega_i} + v|_{\omega_j}).$$  \hfill (2.5)

For each boundary edge-pair $\gamma_k = \{\gamma_k^+, \gamma_k^-, \gamma_k^-\}$, where $\gamma_k^+ \subset \partial \omega_i$, $\gamma_k^- \subset \partial \omega_m$, and $l > m$, we define

$$[v]|_{\gamma_k} = v|_{\omega_l} - v|_{\omega_m}, \quad \langle v \rangle|_{\gamma_k} = \frac{1}{2}(v|_{\omega_l} + v|_{\omega_m}).$$  \hfill (2.6)

III. MODEL PROBLEM AND DG FORMULATIONS

The following periodic boundary value problem is posed over the domain $\Omega$: Find the periodic scalar function $u$ such that

$$- \nabla \cdot (k \nabla u) + cu = f \quad \text{in } \Omega,$$  \hfill (3.1)

where $f \in L^2(\Omega)$ is a known forcing function, and $k, c \in L^\infty(\Omega)$ are prescribed material parameters. Also, we assume that $k(x) > k_0 > 0$ almost everywhere in $\Omega$. 

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To obtain a weak formulation of this problem, suppose that \( u \) is in \( H^2(\mathcal{P}) \). Multiply (3.1) by a smooth function \( v \in H^2(\mathcal{P}) \), integrate over the domain, and decompose the integrals

\[
\sum_{k=1}^{M} \int_{\omega_{Q_k}} (-\nabla \cdot (k \nabla u) + cu) v \, dx = \sum_{k=1}^{M} \int_{\omega_{Q_k}} f v \, dx. \tag{3.2}
\]

Now integrate by parts on each element to obtain

\[
\sum_{k=1}^{M} \int_{\omega_{Q_k}} (k \nabla u \cdot \nabla v + cuv) \, dx - \sum_{k=1}^{M} \int_{\partial\omega_{k}} (k \nabla u \cdot v_k) v \, ds = \sum_{k=1}^{M} \int_{\omega_{Q_k}} f v \, dx, \tag{3.3}
\]

where \( v_k \) is the unit outward normal vector on the element boundary \( \partial\omega_k \). Splitting the boundary integral into terms on the interior edges and terms on the boundary edges, and recalling the earlier definitions of the jump operator,

\[
\sum_{k=1}^{M} \int_{\partial\omega_{k}} (k \nabla u \cdot v_k) v \, ds = \sum_{k=1}^{N_i} \int_{e_k} [(k \nabla u \cdot n) v] \, ds + \sum_{k=1}^{N_b} \int_{\gamma_k} [(k \nabla u \cdot n) v] \, ds, \tag{3.4}
\]

where \( n \) is the unit normal as defined earlier. This can be further simplified to

\[
\sum_{k=1}^{M} \int_{\partial\omega_{k}} (k \nabla u \cdot v_k) v \, ds = \sum_{k=1}^{N_i} \int_{e_k} (k \nabla u \cdot n)[v] \, ds + \sum_{k=1}^{N_b} \int_{\gamma_k} (k \nabla u \cdot n)[v] \, ds. \tag{3.5}
\]

The variational formulation is now given by

\[
\sum_{k=1}^{M} \int_{\omega_{Q_k}} (k \nabla u \cdot \nabla v + cuv) \, dx - \sum_{k=1}^{N_i} \int_{e_k} (k \nabla u \cdot n)[v] \, ds - \sum_{k=1}^{N_b} \int_{\gamma_k} (k \nabla u \cdot n)[v] \, ds = \sum_{k=1}^{M} \int_{\omega_{Q_k}} f v \, dx \quad \forall v \in H^2(\mathcal{P}). \tag{3.6}
\]

On the basis of (3.6), we derive three DG formulations of the problem at hand. In each case we make use of the fact that \([u]\) vanishes due to the assumed smoothness of \( u \).

**The PNIPG Formulation**

Let \( \sigma > 0 \). The PNIPG formulation of the problem is obtained by adding the terms \( \int (k \nabla u \cdot n)[u] \, ds \) to (3.6) on the interior and boundary edges. The discrete formulation reads: Find \( u^h \in V_p \) such that

\[
\sum_{k=1}^{M} \int_{\omega_{Q_k}} (k \nabla u^h \cdot \nabla v + cu^h v) \, dx - \sum_{k=1}^{N_i} \int_{e_k} (k \nabla u^h \cdot n)[v] \, ds - \sum_{k=1}^{N_b} \int_{\gamma_k} (k \nabla u^h \cdot n)[v] \, ds \\
+ \sum_{k=1}^{N_i} \int_{e_k} (k \nabla v \cdot n)[u^h] \, ds + \sum_{k=1}^{N_b} \int_{\gamma_k} (k \nabla v \cdot n)[u^h] \, ds + \sum_{k=1}^{N_i} \int_{e_k} \sigma[u^h][v] \, ds \\
+ \sum_{k=1}^{N_b} \int_{\gamma_k} \sigma[u^h][v] \, ds = \sum_{k=1}^{M} \int_{\omega_{Q_k}} f v \, dx \quad \forall v \in V_p. \tag{3.7}
\]

The terms added on the interior edges $e_k$ enforce weak continuity of the solution across element edges and the terms on the boundary edges $\gamma_k$ enforce weak periodicity.

The PSIPG Formulation

Once again, let $\sigma > 0$. The PSIPG formulation of the problem is obtained by subtracting the terms $\int (k \nabla v \cdot n) [u] \, ds$ from (3.6) on the interior and boundary edges. This makes the formulation symmetric, which reads: Find $u^h \in V^p$ such that

$$
\sum_{k=1}^{M} \int_{\omega_k} (k \nabla u^h \cdot \nabla v + cu^h v) \, dx - \sum_{k=1}^{N_i} \int_{e_k} (k \nabla u^h \cdot n) [v] \, ds - \sum_{k=1}^{N_b} \int_{\gamma_k} (k \nabla u^h \cdot n) [v] \, ds
$$

$$
- \sum_{k=1}^{N_i} \int_{e_k} (k \nabla v \cdot n) [u^h] \, ds - \sum_{k=1}^{N_b} \int_{\gamma_k} (k \nabla v \cdot n) [u^h] \, ds + \sum_{k=1}^{N_i} \int_{e_k} \sigma [u^h] [v] \, ds
$$

$$
+ \sum_{k=1}^{N_b} \int_{\gamma_k} \sigma [u^h] [v] \, ds = \sum_{k=1}^{M} \int_{\omega_k} f \, v \, dx \quad \forall v \in V^p. \quad (3.8)
$$

The PDG Formulation

The PDG formulation of the problem is obtained by setting $\sigma = 0$ in the PNIPG formulation: Find $u^h \in V^p$ such that

$$
\sum_{k=1}^{M} \int_{\omega_k} (k \nabla u^h \cdot \nabla v + cu^h v) \, dx - \sum_{k=1}^{N_i} \int_{e_k} (k \nabla u^h \cdot n) [v] \, ds - \sum_{k=1}^{N_b} \int_{\gamma_k} (k \nabla u^h \cdot n) [v] \, ds
$$

$$
+ \sum_{k=1}^{N_i} \int_{e_k} (k \nabla v \cdot n) [u^h] \, ds + \sum_{k=1}^{N_b} \int_{\gamma_k} (k \nabla v \cdot n) [u^h] \, ds = \sum_{k=1}^{M} \int_{\omega_k} f \, v \, dx \quad \forall v \in V^p. \quad (3.9)
$$

It is clear that each of these three discrete formulations possesses a unique solution. A comparison of the three formulations presented here with the ones discussed in the literature for nonperiodic problems [3, 4, 6] shows that the main difference is in the boundary terms. For nonperiodic problems, typically, one has Dirichlet, Neumann, and possibly mixed boundary conditions imposed on the boundaries. Here, in contrast, we have the requirement that the solution be periodic, and this constraint is incorporated into the variational formulation in a weak manner. As a result, each weak form has additional integral terms defined on the periodic edges.

A Priori Error Estimates. Because the formulations presented here only amount to a special treatment of the periodic edges, it is quite straightforward to show that the a priori error estimates developed for the nonperiodic versions [3, 4, 8, 9] are also applicable here. We thus have the following propositions.

Proposition 3.1 (PNIPG Formulation). Let $u^h$ be the discrete solution to (3.7) with $\sigma = \kappa / h$, $\kappa > 0$ and let $u \in H^1(\Omega) \cap H^s(\mathcal{P})$, $s \geq 2$. Then

$$
\| u - u^h \|_{1,p} \leq C \frac{h^{s-1}}{p^{s-3/2}} \| u \|_s
$$

(3.10)
where $\mu = \min(p + 1, s)$, and $C$ is a constant independent of $h$ and $p$. Furthermore,

$$\|u - u_h\|_{0,P} \leq Ch^{\mu-1}\|u\|_s$$

(3.11)

with $\mu$ as before and $C$ independent of $h$.

**Proposition 3.2 (PSIPG Formulation).** Let $u_h$ be the discrete solution to (3.8) with $\sigma = \kappa/h$, $\kappa \geq \kappa_0 > 0$ and let $u \in H^1 \cap H^s(P), s \geq 2$. Then

$$\|u - u_h\|_{1,P} \leq C \frac{h^{\mu-1}}{p^{\gamma(3/2)}}\|u\|_s$$

(3.12)

where $\mu = \min(p + 1, s)$, and $C$ is a constant independent of $h$ and $p$. Moreover,

$$\|u - u_h\|_{0,P} \leq Ch^{\mu}\|u\|_s,$$

(3.13)

with $\mu$ as before and $C$ independent of $h$.

**Proposition 3.3 (PDG Formulation).** Let $u_h$ be the discrete solution to (3.9) with $p \geq 2$ and let $u \in H^1 \cap H^s(P), s \geq 2$. Then

$$\|u - u_h\|_{1,P} \leq C \frac{h^{\mu-1}}{p^{\gamma-(5/2)}}\|u\|_s,$$

(3.14)

where $\mu = \min(p + 1, s)$. Further,

$$\|u - u_h\|_{0,P} \leq Ch^{\mu-1}\|u\|_s,$$

(3.15)

with $\mu$ as before and $C$ independent of $h$.

Note that the $L^2$ estimates for the PNIPG and PDG formulations are suboptimal in $h$, whereas the one for PSIPG is optimal.

**IV. NUMERICAL EXAMPLE**

Consider the following boundary value problem posed on a square domain $\Omega = (0, 1)^2$: Find a periodic function $u$ such that $\int_\Omega u \, dx = 0$ and

$$-\nabla \cdot (k \nabla u) = 8\pi^2 \sin(2\pi x) \sin(2\pi y) \quad \text{in } \Omega.$$  

(4.1)

With $k = 1$, the exact solution for the problem is simply

$$u = \sin(2\pi x) \sin(2\pi y).$$

(4.2)

We solve this periodic boundary value problem with the PNIPG, PSIPG, and PDG formulations. For each case, we use a uniform mesh with $h = 1/2, 1/4, 1/8, 1/16, 1/32$, and $1/64$. For
FIG. 2. Convergence for the PNIPG formulation with $\sigma = 1/h$: (a) $H^1$ norm convergence and (b) $L^2$ norm convergence.

each mesh size, we use $p = 1, 2, \text{and } 3$ for the PNIPG and PSIPG formulations. For the PDG formulation, we use $p = 2$ and 3.

The numerical convergence rates in the $H^1$ and $L^2$ norms for the PNIPG formulation are shown in Fig. 2. The penalty parameter is taken to be $\sigma = 1/h$. The convergence rates are as expected from Proposition 3.1.

The numerical convergence rates for the PSIPG and PDG formulations are shown in Figs. 3 and 4, respectively. For the PSIPG case, a penalty of $10/h$ was necessary to obtain optimal convergence rates. Again, the convergence rates are as predicted by theory.
FIG. 3. Convergence for the PSIPG formulation with $\sigma = 10/h$: (a) $H^1$ norm convergence and (b) $L^2$ norm convergence.

V. CONCLUDING REMARKS

The solution of periodic elliptic boundary value problems using discontinuous Galerkin methods is discussed in this article. The periodicity boundary condition is imposed weakly by penalizing the jump in the solution across periodic edges, much like the jump in the solution across interior element edges. *A priori* estimates are presented for the various DG formulations and are confirmed numerically for a simple two-dimensional problem. We find that the methods behave essentially like their nonperiodic counterparts in terms of both $H^1$ and $L^2$ convergence rates.
One area of interest for future work is the extension of these formulations to nonscalar problems, e.g., linear elasticity. The application of the proposed formulations to problems of periodic composites and to problems in structural topology optimization is currently under investigation.

References


