X-Ray Computerized Tomography: A Review

Kumar S. Venaganti

CAE Research Laboratory, Department of Mechanical, Industrial and Nuclear Engineering
University of Cincinnati
Cincinnati, Ohio, U.S.A.
E-mail: Kumar.Venaganti@uc.edu

In this note, we review the basic theory of X-ray Computerized Tomography. Following a discussion of some important properties of X-rays, the essentials of tomography are presented using elementary concepts from signal processing.

Key Words: computerized tomography; X-rays

1. INTRODUCTION

Tomography refers to the cross-sectional imaging of an object from data collected by subjecting the object to electro-magnetic radiation from different directions. It is a form of Non-Destructive Analysis (NDA) that has been used in such diverse areas as diagnostic medicine, mapping of underground resources as well as the study of celestial objects [1]. X-ray computerized tomography refers to the use of X-rays to analyze a given cross-section.

One application of X-ray computerized tomography that is of particular interest to the mechanics community is the determination of microstructure of composite materials, i.e., the generation of cross-sectional images of the material density distributions or material maps [2, 4]. Our goal in this paper is to summarize and succinctly describe the basic theory of CT as applied to the generation of material maps. Since this paper is geared towards readers who may be unfamiliar with tomography and signal processing, the treatment is rudimentary and advanced topics are not addressed. Also, where necessary, sufficient regularity of functions is assumed. For an advanced treatment of the subject, the reader is referred to [1] and the references cited therein. For a mathematical treatment, see [3]. To a great extent, this paper follows [1], including the notation.

Following this short introduction, we discuss the generation and properties of X-rays in Section 2. Section 3 deals with tomography, and details the generation of material maps of objects.
2. GENERATION AND PROPERTIES OF X-RAYS

X-Rays are a form of electro-magnetic radiation of short wavelengths (10^{-8} to 10^{-11} meters) discovered by W.C. Röntgen in 1895. X-rays are produced when high-speed electrons strike a solid target and their kinetic energy is converted into energy packets or radiation quanta. The wavelength of the radiation depends on the energy of the electrons. Typically, a very high voltage is applied between the electrodes of an X-ray tube that has been evacuated to a low pressure (≈ 1μm of mercury) causing electrons to flow between the electrodes. The energy of the electrons depends on the applied voltage and the pressure in the X-ray tube. The electrodes are arranged such that the resulting flow of electrons strikes a metal target resulting in the release of photons. Typical metal targets include Molybdenum (Mo), Tungsten (W), Copper (Cu) and Chromium (Cr). A schematic of an X-ray tube is shown in Figure 1(a).

One of the most important properties of X-rays, used to detect their presence, is their ability to ionize gases. On striking a gas molecule, an X-ray photon produces a burst of electrons and ions, resulting in a current pulse. An X-ray detector device usually consists of a gas-filled cylindrical chamber in which an electric field is maintained, with the wall of the cylinder acting as the cathode and a wire passing along the axis of the cylinder acting as the anode. A schematic of an ionization chamber is shown in Figure 1(b). X-ray photons enter the chamber and strike gas molecules resulting in the release of positive ions and electrons. Under the influence of the electric field, the ions move towards the cathode and the electrons move towards the anode. This movement of charged particles causes an observable flow of a current pulse in the circuit joining the electrodes. The magnitude of this pulse is a measure of the intensity of the X-ray, and is denoted by the symbol I.

![Schematic representation of (a) an X-ray tube and (b) an Ionization chamber used to detect X-rays.](image)

Another important property is that an X-ray beam, on passing through any material, undergoes attenuation - i.e. it experiences a reduction in its intensity I. The two mechanisms responsible for the attenuation are the Photoelectric absorption effect and the Compton effect. Photoelectric absorption refers to the absorption of
a photon’s energy by an inner electron of a material atom. Some of the photon’s energy is used to overcome the binding energy within the electron’s shell and the rest appears as kinetic energy. The Compton effect refers to the interaction of a free electron or an outer electron (with low binding energy) with a photon. This interaction leads to a deflection of the photon accompanied by a loss of energy. This energy is imparted to the electron in the form of kinetic energy.

The attenuation of an X-ray beam is mathematically expressed by relating the intensity of the beam traveling through a material to the intensity of the incident beam. For this purpose, consider a slab of the material of infinitesimal thickness ds upon which a beam of intensity \( I \) is incident. The change in the intensity of the beam \( dI \) in traveling through \( ds \) is given by [1]

\[
\frac{dI}{I} = -\mu ds,
\]

where \( \mu > 0 \) is the linear attenuation coefficient of the material. The linear attenuation coefficient accounts for the overall attenuation of the X-ray beam due to the Photoelectric and Compton effects. If the material is homogeneous, we obtain upon integration

\[
I(s) = I_0 \exp(-\mu s),
\]

where \( s \) is the distance traveled by the beam through the material and \( I(s) \) is the intensity of the beam as a function of position, and \( I_0 \) is the intensity of the incident beam (at \( s = 0 \)). If the material is inhomogeneous, then the attenuation coefficient \( \mu \) is a function of position \( (x,y) \). We then have

\[
\int_0^s \mu(s(x,y)) \, ds = \ln \left( \frac{I_0}{I(s)} \right),
\]

and if we consider the entire path of the beam through the material, we obtain

\[
\int_{\text{Beam Path}} \mu(s(x,y)) \, ds = \ln \left( \frac{I_0}{I_{\text{exit}}} \right),
\]

where \( I_{\text{exit}} \) is the intensity of the beam upon exiting the material. This equation is fundamental to computerized tomography in that the integral on the left hand side can be evaluated simply by knowing the intensity of the X-ray beam before and after it passes through the material. It will be seen shortly that by knowing the integral on the left hand side of (4) for beams of several different orientations, it is possible to reconstruct the function \( \mu(x,y) \). Then, knowing the relationship between the attenuation coefficient and material density, the material map of the cross-section can be constructed. Therefore, from this point on, we restrict our attention to the reconstruction of the function \( \mu(x,y) \).

3. MATERIAL MAP GENERATION USING TOMOGRAPHY

Let \( \omega \subset \mathbb{R}^2 \) be a open bounded domain as shown in Figure 2. We are interested in reconstructing the function \( \mu(x,y) : \omega \to \mathbb{R} \), where \( (x,y) \) is an orthogonal coordinate system on \( \mathbb{R}^2 \).
A second orthogonal coordinate system \((r, s)\) is obtained by rotating the \((x, y)\) coordinate system by an angle \(\theta\)

\[
\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]

(5)

A line in \(\mathbb{R}^2\) that is inclined at an angle \(\pi/2 + \theta\) to the \(x\)-axis, i.e., parallel to the \(s\)-axis, can be specified by its \(r\) coordinate and its equation is given by \(x\cos \theta + y\sin \theta = r\). This line is denoted \(L(r, \theta)\). The line integral of the function \(\mu\) along line \(L(r, \theta)\) is defined as:

\[
P(r, \theta) \overset{\text{def}}{=} \int_{L(r, \theta)} \mu(r, s) \, ds = \int_{s \in \mathbb{R}} \mu(r, s) \, ds,
\]

(6)

with the understanding that \(\mu\) is identically equal to zero for points on the line \(L(r, \theta)\) that are outside \(\omega\). By computing the line integral of \(\mu\) along lines \(L(r, \theta_0)\) for a fixed \(\theta_0\) while varying \(r\), we obtain a parallel projection, as shown in Figure 3. Note that the projection at an angle \(\theta_0\), also denoted by \(P\), is a function of \(r\) alone.

For now, we assume that the CT device measures the projection \(P(r, \theta_0)\) for various \(\theta_0\) using (4). The key mathematical tool in reconstructing \(\mu(x, y)\) from the projections at various angles is the Fourier Slice theorem [1] which can be stated as follows:

**Theorem 3.1** (Fourier Slice Theorem). Let \(\hat{\mu}(u, v)\) be the two-dimensional Fourier transform of \(\mu\), and let \(\hat{P}(\xi)\) be the one-dimensional Fourier transform of \(P\) for a given \(\theta_0\). Then, \(\hat{P}\) gives the values of \(\hat{\mu}\) along a line inclined at an angle \(\theta_0\) to the \(u\)-axis in the Fourier space, i.e.,

\[
\hat{\mu}(u, v = u \tan \theta_0) = \hat{P}(\xi)
\]

(7)

for \(\xi = u \cos \theta_0 + v \sin \theta_0\).
\[ \hat{\mu}(u, v) = \int_{\mathbb{R}^2} \mu(x, y) \exp\left(-i(xu + yv)\right) \, dx \, dy, \]  

(8)

and using the coordinate transformation (5),

\[ \hat{\mu}(u, v) = \int_{\mathbb{R}^2} \mu(r, s) \exp\left(-i(r(u \cos \theta_0 + v \sin \theta_0) - s(u \sin \theta_0 - v \cos \theta_0))\right) \, dr \, ds. \]  

(9)

On the line \( v = u \tan \theta_0 \), we have \( u \sin \theta_0 = v \cos \theta_0 \). Hence,

\[
\hat{\mu}(u, v = u \tan \theta_0) = \int_{r \in \mathbb{R}} \mu(r, s) \exp\left(-i(r u \cos \theta_0 + v \sin \theta_0)\right) \, dr \, ds \\
= \int_{r \in \mathbb{R}} \left[ \int_{s \in \mathbb{R}} \mu(r, s) \, ds \right] \exp(-ir \xi) \, dr,
\]

where \( \xi \overset{\text{def}}{=} u \cos \theta_0 + v \sin \theta_0 \) is the coordinate along the line \( v = u \tan \theta_0 \) in the Fourier plane. Recognizing that

\[ \int_{s \in \mathbb{R}} \mu(r, s) \, ds = P(r, \theta_0), \]  

(11)

we obtain

\[
\hat{\mu}(u, v = u \tan \theta_0) = \int_{r \in \mathbb{R}} P(r, \theta_0) \exp(-ir \xi) \, dr \\
= P(\xi),
\]

(12)

for \( \xi = u \cos \theta_0 + v \sin \theta_0 \), which completes the proof. \( \square \)

**Proof.** By definition,

**FIG. 3.** A parallel projection is obtained by taking a series of line integrals with \( \theta = \theta_0 \) fixed and \( r \) varying.
in principle, by computing the Fourier transforms of projections $P(r, \theta_0)$ for an infinite number of angles $\theta_0$, the function $\hat{\mu}$ is completely known (at each point in the Fourier plane). Next, simply by taking the inverse Fourier transform of $\hat{\mu}$, the function $\mu$ can be reconstructed.

![Diagram of Fourier Slice Theorem](image)

**FIG. 4.** Illustration of the Fourier Slice Theorem.

### 3.1. The Filtered Backprojection Algorithm

With the aid of the Fourier Slice theorem, it is possible to reconstruct the function $\mu(x, y)$ exactly provided projections $P(r, \theta_0)$ are taken for an infinite number of values of $\theta_0$. In practice, however, projections are taken at only a finite number of orientations. Also, the procedure discussed in the previous subsection does not
lend itself to easy implementation on a computer owing to the fact that the use
of discrete Fourier forward/inverse transforms requires data on a rectangular grid
as opposed to a set of radial lines. Thus, a different approach is required for
practical implementations. The most widely used method for this is the Filtered
Backprojection algorithm. We now briefly describe this algorithm.

First, by definition,

$$\mu(x, y) = \int_{\mathbb{R}^2} \hat{\mu}(u, v) \exp(i(ux + vy)) \, dx \, dy.$$  (13)

We introduce a polar coordinate system \((w, \theta)\) in the Fourier plane such that
\(u = w \cos \theta\) and \(v = w \sin \theta\). Now the function \(\mu\) can be expressed as

$$\mu(x, y) = \int_0^{2\pi} \int_0^\infty \hat{\mu}(w, \theta) \exp(iw(x \cos \theta + y \sin \theta)) \, dw \, d\theta.$$  (14)

This can be written as

$$\mu(x, y) = \int_0^\pi \int_0^\infty \hat{\mu}(w, \theta) \exp(iw(x \cos \theta + y \sin \theta)) \, dw \, d\theta$$
$$+ \int_0^\pi \int_0^\infty \hat{\mu}(w, \theta + \pi) \exp(iw(x \cos(\theta + \pi) + y \sin(\theta + \pi))) \, dw \, d\theta,$$

and since \(\hat{\mu}(w, \theta + \pi) = \hat{\mu}(-w, \theta)\), we obtain

$$\mu(x, y) = \int_0^\pi \int_{-\infty}^{\infty} \hat{\mu}(w, \theta) \exp(iw(x \cos \theta + y \sin \theta)) \, dw \, d\theta,$$
$$= \int_0^\pi \int_{-\infty}^{\infty} \hat{P}(w) \exp(iw(x \cos \theta + y \sin \theta)) \, dw \, d\theta,$$

(16)

since by the Fourier Slice theorem,

$$\hat{\mu}(w, \theta) = \hat{\mu}(u = w \cos \theta, v = w \sin \theta) = \hat{P}(w(\cos^2 \theta + \sin^2 \theta)).$$  (17)

This is now written as

$$\mu(x, y) = \int_0^\pi Q(x \cos \theta + y \sin \theta) \, d\theta,$$  (18)

where

$$Q(x \cos \theta + y \sin \theta) \overset{\text{def}}{=} \int_{-\infty}^{\infty} \hat{P}(w) \exp(iw(x \cos \theta + y \sin \theta)) \, |w| \, dw.$$  (19)

Equations (19) and (18) are referred to as the filtering and backprojection steps,
respectively. For an explanation of these terms, see [1]. If it is now assumed that
\(\hat{P}\) is band-limited with bandwidth \(W\), i.e., \(\hat{P}(w) = 0\) for \(w > W\), then we can write

$$Q(x \cos \theta + y \sin \theta) = \int_{-W}^{W} \hat{P}(w) \exp(iw(x \cos \theta + y \sin \theta)) \, |w| \, dw.$$  (20)
Finally, in a discrete form, these two expressions are written as:

\[
Q(t) \approx \frac{2W}{N} \sum_{m=-N/2}^{N/2} \hat{P}
\left( m \frac{2W}{N} \right) \left| m \frac{2W}{N} \right| \exp\left( i2\pi m \left( \frac{2W}{N} \right) t \right)
\]

(21)

for some large \( N \), and

\[
\mu(x, y) \approx \frac{\pi}{K} \sum_{j=1}^{K} Q(x \cos \theta_j + y \sin \theta_j).
\]

(22)

Using the above two expressions and using the Fast Fourier Transform (FFT), the function \( \mu(x, y) \) can now be reconstructed. Note that it might sometimes be necessary to use interpolation to compute \( \hat{P} \) for a given value of \( m \). Also, it might be necessary to apply a filter to reduce high-frequency noise in the computation of \( Q \). Further details on the filtered back-projection algorithm can be found in [1].

4. CLOSING REMARKS

This note reviews some of the fundamentals of X-ray computerized tomography. The interested reader may consult [1] or [3] for an advanced treatment of this subject.

REFERENCES